Parvatibai Chowgule College of Arts and Science
Autonomous

BSc. Semester End Examination, January/February 2022

Semester: III
Subject: Mathematics
Title: Number Theory I (Elective)
Duration: 2 hours Max Marks: 60

Instructions: 1. All questions are compulsory. However internal choice is applicable.
2. Figures to the right indicate full marks.
3. **Justify all responses.**

Q1. Answer <u>ANY 3</u> of the following: **(12)**

a) Show that an integer which is simultaneously a square and a cube is either of the form $7k$ or $7k + 1$.

b) What is $\gcd(a^2 + b^2, a + b)$, where a and b are relatively prime integers, that are not both zero? Justify your answer.

c) Define relatively prime integers and find the number of integers ≤ 21000, which are not relatively prime to 21000.

d) Solve the linear congruence $9x \equiv 12 \ (mod\ 15)$

Q2. Answer <u>ANY 3</u> of the following: **(12)**

a) Prove that $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$ for each positive integer n.

b) Find the exponent of 6 in 553!

c) Show that 45 is a pseudoprime to the bases 17 and 19.

d) Show that if a and b are positive real numbers, then $[ab] \geq [a][b]$. What is the corresponding inequality when both a and b are negative? When one is negative and the other positive?

Q3. Answer the following: **(12)**

a) Solve the following system of congruences

$$2x \equiv 1 \ (mod\ 5)$$
$$3x \equiv 9 \ (mod\ 6)$$
$$4x \equiv 1 \ (mod\ 7)$$
$$5x \equiv 9 \ (mod\ 11)$$

b) A shopkeeper purchased 100 stationary items for a total cost of Rs. 4000. Prices were as follows: books Rs. 120 each, crayon boxes Rs. 50 each, pens Rs. 25 each.

If the shopkeeper obtained at least one item of each type, how many of each did he buy?

Q4. Answer <u>ANY 2</u> of the following: **(12)**

a) Let p be prime and $\gcd(a, p) = 1.$ By using Fermat's theorem solve the linear congruence $ax \equiv b\ (mod\ p)$. Also generalize it when p is not prime.

b) Prove that an integer n is prime if and only if $(n-1)! \equiv -1\ (mod\ n)$

c) Let k be an integer ≥ 2. Prove that

(i) $\sigma(2^{k-1}) = 2^k - 1$

(ii) If $2^k - 1$ is a prime and $n = 2^{k-1}(2^k - 1)$, then $\sigma(n) = 2n.$

(iii) If $2^k - 3$ is a prime and $n = 2^{k-1}(2^k - 3)$, then $\sigma(n) = 2n + 2.$

Q5. Answer <u>ANY 2</u> of the following: **(12)**

a) Let $n = p_1^{k_1} p_2^{k_2} \dots p_t^{k_t}$ be the prime factorization of a positive integer n. Two functions ρ and λ are defined as follows:

(i) $\rho(n) = \begin{cases} 1 & if\ n = 1 \\ 2^t & otherwise \end{cases}$

(ii) $\lambda(n) = \begin{cases} 1 & if\ n = 1 \\ (-1)^{k_1 + k_2 + \cdots + k_t} & otherwise \end{cases}$

Prove that ρ and λ are both multiplicative functions.

b) Find all primitive Pythagorean triples of the form x, y, z when $x = 60\ and\ x = 80$

c) Solve the system of linear congruences

$$2x + 3y \equiv 5\ \ (mod\ 7)$$

$$x + 5y \equiv 6\ (mod\ 7)$$

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